Complex topology in chaotic scattering: a laboratory observation

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In chaotic scattering, an initially freely-moving orbit (e.g. that of an atom or star) enters a scattering region and evolves chaotically for some period of time, before escaping and returning to free motion [1]. We are interested in cases where escape can occur in three or more distinct ways. Here we demonstrate, using a laboratory model, that the regions of state space (called basins) corresponding to different ways of escaping can have an interesting topological property which we call the *Wada property*, by which we mean that these regions may be so convoluted that every point on the boundary of a basin is on the boundary of all basins [2, 3].

General interest in chaotic scattering stems from its ability to model complex phonemena in a variety of physical systems, including chemical reactions[4], celestial mechanics[5], fluid dynamics[6], and electron scattering in semiconductor microstructures[7]. In our laboratory model [8] we consider the path of a light ray as it reflects from a scatterer consisting of four mirrored balls arranged as in Fig. 1. A light ray entering the "inner chamber" between the four balls can exit through one of four openings (Figs. 1a and 1b). Thus we can identify four basins, each consisting of the collection of light rays which exit through one of the four openings. The boundary separating these basins has the Wada property. This property is of interest in chaotic scattering situations because it impedes one's ability to predict from an initial condition the escape mode (for our model, the opening through which a light ray will exit the scatterer).

That a division of space into three or more regions can have such a strange topology was shown in 1917 by the Japanese mathematician Wada and in 1910 by the Dutch mathematician L. E. J. Brouwer [9]. The mathematicians Julia and Fatou proved, in 1918 and 1920, respectively, a theorem which implies that the boundary separating the basins (in the complex plane) of the four roots found using Newton's method for a fourth-degree polynomial has the topology of the basin boundary we examine here [10].

The photograph in Fig. 2 shows a view looking in through one of the exits of the scatterer. Two of the colors, red and blue, are the reflections of poster boards placed outside two of the exits. Light reflects from a poster board, then, having taken the board's color, it reflects off the spheres, perhaps many times, before entering the camera lens. (Similar effects create the white patches. We passed light through a white diffusive filter before allowing it to pass into the scatterer.) The black areas result from leaving the fourth exit uncovered and the room dark. Had light been shone on the camera and photographer, we would have seen their reflections in the black areas.

In this picture, the four colors create a map of the four basins. If an observer were to aim a narrow beam of light from his/her eye to a blue patch, for instance, that beam would, after some reflections, hit the blue poster board. It would leave the scatterer through the "blue" opening and be said to have started in the blue basin. The boundary between the four colors is a fractal, and its fractal dimension, determined from numerical simulation, is approximately $D = 1.6$. Thus, our work provides a simple laboratory demonstration of an interesting type of fractal geometry.

To numerically study this system we used the geometrical optics approximation (i.e., diffraction is neglected) in which the trajectory of a light ray reflecting from the spheres corresponds to that of a point particle bouncing off of the spheres. We plotted a picture similar to that in Fig. 2 (see Fig. 1c) and magnified regions containing points in the basin boundary (e.g., Fig. 1d). We find that no matter where – or by how much – we magnify around a boundary point we always see all four basin colors. Thus arbitrarily close to any boundary point we find points in all four basins, and every boundary point is on the boundary of all four basins.

Imagine performing an experiment on a scattering system with the Wada property. We would measure the initial conditions within some error bounds and record the outcome of the experiment. If the initial condition were sufficiently near the basin boundary, then the small, random, error in the initial condition could push the experiment into any of the possible outcomes, because of the Wada property.

We have experimentally verified the Wada property for our laboratory model using a 0.48 mm diameter beam from a 0.95 mW HeNe laser. If the beam is shone on a boundary point we see that the laser light can be seen through each of the openings. On the other hand, if the beam is aimed at the interior of one of the colored regions, the light is seen through only one opening, casting a bright spot on the corresponding poster board.

We believe that Wada boundaries should be a typical feature in chaotic scattering systems which have more than two exit modes. In support of this, we remark that we have verified the existence of Wada boundaries (in a calculation to be reported elsewhere) in a situation involving chemical reactions [4] with two and three degrees of freedom.

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Figure Legends

Figure 1: The scattering system. Four mirrored spheres (approximately 15 cm radius) are stacked like cannonballs, three resting on the table and the fourth placed on top of the three so that they are all touching. (a) A schematic of the scatterer. The dashed/dotted lines show how the sphere centers are located at the vertices of a tetrahedron (pyramid). A light ray can enter/exit the inner chamber of the scatterer through one of four openings lying in the four faces of the tetrahedron. Once a light ray exits it cannot return to the inner chamber. (b) Schematic of the shape (shaded) of one of the openings on a tetrahedron face. (c) Plot of basins created using numerical simulation. Light rays are initiated on a portion of the surface of one of the spheres which faces into the scatterer. Each light ray is aimed away from the viewer. (d) If a small diameter laser beam were shown on the circled region in (c), parts of the beam would land in each of the four basins, as indicated by the presence of all four colors in this blow-up of the circled region. Blowups of successively smaller sections of the boundary always contain all four colors. This suggests that the basins are Wada.

Figure 2: Photograph of the laboratory model of the scatterer described in Fig. 1. Red and blue posterboards were placed outside of two of the openings and a white cloth below. Their reflections create the colored pattern. The boundary separating the colors has a fractal dimension of approximately 1.6.

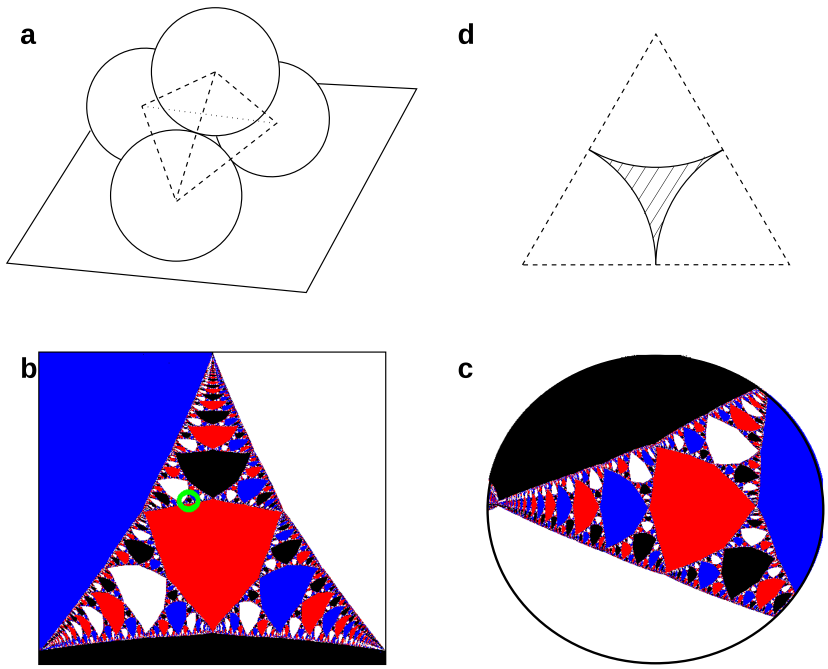

Figure 1:

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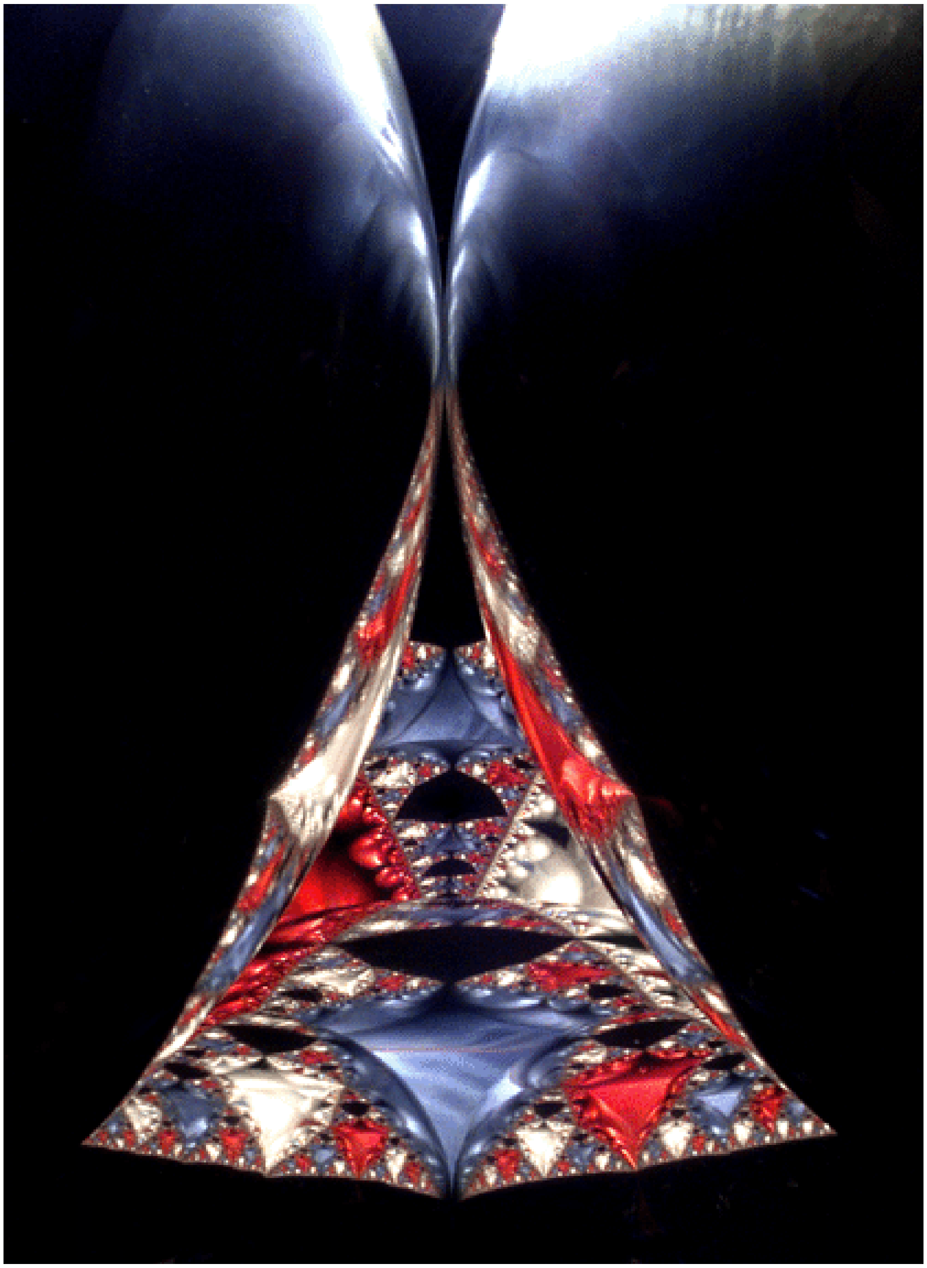

Figure 2:

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