Fractal Basin Boundaries in Higher-Dimensional Chaotic Scattering

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We analyze a hard-walled billiard chaotic scattering system in three spatial dimension. Our analysis of this system tests a conjectured formula for the fractal dimension of "typical" non-attracting chaotic sets in higher-dimensional systems (e.g., time-independent, Hamiltonian systems with more than two degrees of freedom). It also shows the occurrence, in a chaotic scattering system, of a fractal basin boundary whose structure is that of a continuous, nowhere differentiable surface. A ray optical experimental realization of the billiard is suggested, and would offer the possibility of a physical realization of this basic type of basin boundary structure.

Fractal geometry is a fundamental attribute associated with chaos in a variety of situations. The most well-known is the common occurrence of fractal *attractors* [the sets in phase space to which orbits tend with time are called attractors], and such fractal sets have also been called "strange attractors." In addition to chaotic attractors, nonattracting chaotic sets (also called chaotic repellors) are also of great practical interest. [A nonattracting chaotic set is one for which orbits placed precisely on the set move around chaotically, and for which orbits placed near the set (but not precisely on it) are repelled from the set and move away.] In particular, nonattracting chaotic sets arise in the consideration of chaotic scattering, and fractal basin boundaries [1]. [A basin is a region of state space such that orbits from initial conditions in that basin yield a particular outcome; e.g., they all go to a particular attractor, or, in the case of scattering, they all are scattered in a particular way (in our subsequent example, "scattered up" or "scattered down").] In these situations the fractal nature of the nonattracting chaotic set leads to difficulty in prediction of the outcomes resulting from initial conditions (this is one of the basic challenges to determinism from the existence of chaos). For example, in the case of fractal basin boundaries the probable error in the prediction of an outcome typically scales as a power in the uncertainty with which the initial condition is known, and the power law exponent is quantitatively given by the dimension of the state space minus the fractal dimension of the boundary [1].

In this paper we discuss a nonattracting chaotic set resulting from a particular chaotic scattering problem. This example is of interest from several points of view:

(1) It provides an example illuminating the applicability of a conjectured formula [2,3] giving the fractal dimension of higher-dimensional nonattracting chaotic sets in terms of their Lyapunov exponents and exponential escape times (defined subsequently). (This formula has not been previously tested by numerical experiments.)

(2) The example reveals a new structure for invariant sets of chaotic scattering systems with more than two degrees of freedom. In particular, we find fractal basin boundaries that are continuous surfaces that are not smooth; they are "wrinkled" on arbitrarily fine scale (nondifferentiable) [4].

(3) As discussed at the end of the paper, the example appears to be readily amenable to an optical experimental realization (in which the orbits are light rays). In such a realization the stable manifold and its fractal structure are in principle observable "by eye." (In general, experiments observing the structure of fractal basin boundaries [5,6] are rare due to the inherent difficulties they present. These difficulties are absent in our suggested realization.)

The system studied is shown in Fig. 1. An ellipsoid is placed in an infinitely long (in z) tube with cross section as shown in Fig. 1(b). We consider a freely moving point particle experiencing energy-conserving, specular reflection from the walls of the tube and the surface of the ellipsoid. We shall consider two cases: (i) the major axis of the ellipsoid is perpendicular to the $x-y$ plane, and (ii) the major axis of the ellipsoid is tilted with respect to the z-axis and lies in the $y-z$ plane. In both cases the center of the ellipsoid is located in the center of the tube (the origin).

We construct a map by recording the particle position (we use cylindrical coordinates, (z, ϕ)) and direction (v_z, v_ϕ) at each bounce from the ellipsoid. (We set $|\vec{v}| = 1$ since energy is conserved.) After the particle passes the top of the ellipsoid with $v_z > 0$ or the bottom with $v_z < 0$, it continues toward infinity. When this occurs we say that the particle has escaped.

We begin by considering case (i) where the major axis of the ellipsoid is vertical. Orbits of the map started in the manifold $z = 0, v_z = 0$ (denoted Λ) never leave it but bounce around chaotically in the ϕ and v_ϕ coordinates. In Λ, the particle sees the two dimensional billiard shown in Fig. 1(b). This billiard has the property that almost every orbit comes arbitrarily close to any point in its phase space. Thus, Λ is a two dimensional, ergodic, invariant set. For typical orbits with respect to the natural measure on the invariant set [7], this system has a Lyapunov exponent pair $\pm h_\phi$ characterizing motion in Λ and another, $\pm h_z$, characterizing motion toward or away from Λ.

When the ellipsoid is given a small tilt we find that the ergodic, chaotic, invariant set, Λ, changes its character from a two dimensional planar set to a set with fractal dimension greater than two. In either case (i.e., tilted or untilted) the stable manifold of Λ (which we denote SM) is of particular physical importance in that it divides the space of initial conditions into two regions (basins), one yielding orbits that eventually escape to $z \to +\infty$ and the other, orbits that escape to $z \to -\infty$.

Figures 2(a) and 2(b), corresponding to the untilted and tilted cases, respectively, show two dimensional cross sections of the full phase space of the basins of $z \to \infty$ (white) and $z \to -\infty$ (black). We subsequently present evidence that the boundary between black and white appears to be a continuous, nowhere differentiable curve. Points on the boundary do not escape but, rather, asymptote to Λ (i.e., they are on SM).

In the following, when we numerically consider the case of the tilted ellipsoid, we shall consider the tilt angle to be small. In this case the Lyapunov exponents for typical orbits with respect to the natural measure [7] on Λ are approximately unchanged from their values obtained in the case of zero tilt, and we continue to denote them by h_z and h_ϕ. Considering a chaotic scattering system with Lyapunov exponents $\pm h_z$ and $\pm h_\phi$, with respect to the natural measure, the result of [2] and [3] is that for typical systems the dimension (information dimension of the measure [7]) of SM is

$$D = 4 - (h_* \tau)^{-1}, \tag{1}$$

when $h_* \tau \geq 1$, where $h_* = max(h_z, h_\phi)$. In (1) the quantity τ is the exponential escape time from Λ defined as follows. Imagine that we sprinkle a cloud of initial conditions in a region including the nonattracting chaotic set Λ. As time increases almost all of them eventually leave the vicinity of the set, and there is a characteristic escape time, τ, such that, at late time, the fraction of the cloud still in the vicinity of the set decays with time t as $\exp(-t/\tau)$. For the case we consider, we find that $h_\phi > h_z$ for $h_* \tau \geq 1$, and, thus, $h_* = h_\phi$ in (1).

We find that Eq. (1) holds in the tilted case, but not in the untilted case. The fact that (1) is violated in the untilted case is related to the original conjecture [2,3] which (like the Kaplan-Yorke conjecture [8] for chaotic attractors) claims that the given dimension formulae [in this case Eq. (1)] only apply for "typical" systems. Thus, the issue of what constitutes a typical system is central to the determination of dimension from Lyapunov exponents. At present, there is no rigorous formulation of typical for this purpose. Thus, it is important to address this question through examples, such as the present billiard system [9].

When the ellipsoid is not tilted, analysis given subsequently shows that the SM dimension is [9]

$$D = 4 - (h_z + \tau^{-1})/h_\phi \text{ for } h_\phi \geq h_z + \tau^{-1}. \tag{2}$$

Since this formula is valid only in the special case where the ellipsoid has no tilt, we call the untilted ellipsoid scattering system *atypical*. [In the case of very small tilt, $lnN(\varepsilon)$ scales linearly with $ln(1/\varepsilon)$ with slope given by (2) for $\varepsilon \stackrel{\sim}{>} \varepsilon_*$ and subsequently, for $\varepsilon \stackrel{\sim}{<} \varepsilon_*$, crosses over to slope given by (1) (here ε_* is a small tilt-dependent crossover value). In such a case the dimension, which is defined for the $\varepsilon \to 0$ limit, is that given by (1).]

The dimension of the stable manifold of Λ for these systems was calculated numerically for several values of $h_\phi \tau$. In these calculations $h_\phi \tau$ was varied by changing the height of the ellipsoid (while the width of the ellipsoid remained fixed). The box-counting dimension of SM was computed using the uncertainty dimension technique [1,10,11] . This technique gives the dimension of the basin boundary, but, as discussed above, the stable manifold forms the basin boundary. As shown in Fig. 3, the results from the computation agree reasonably well with (1) and (2). The box-counting dimension is an upper bound to the information dimension. In examples where the two can be calculated (e.g., for attractors) their values commonly turn out to be very close. Since SM divides the space, its box-counting dimension is at least three. Thus, consistent with the data of Fig. 3, we expect that the box-counting dimension is three where $(h_\phi \tau)^{-1} > 1$ ($h_\phi > h_z + \tau$) for the tilted (untilted) case.

To illustrate the dimension formulae and the possible existence, for our chaotic scatterer, of a basin boundary structure in the form of a continuous nowhere differentiable surface, we consider the untilted case. In this case the essential property implied by the symmetry of the untilted configuration is that, if $z = v_z = 0$ initially, then $z = v_z = 0$ for all time. Let $\vec{z} = (z, v_z)$ and $\vec{\phi} = (\phi, v_\phi)$. For points near Λ (i.e., $|\vec{z}| \ll 1$) we can expand the map function in $\vec{z}$ to obtain

$$\vec{z}_{n+1} = DM_z(\vec{\phi}_n)\vec{z}_n + \delta\vec{M}_z(\vec{z}_n, \vec{\phi}_n)$$
$$\vec{\phi}_{n+1} = \vec{M}_\phi(\vec{\phi}_n) + \delta\vec{M}_\phi(\vec{z}_n, \vec{\phi}_n).$$

$DM_z(\vec{\phi})$ is a 2x2 matrix. $DM_z(\vec{\phi})\vec{z}$ has Lyapunov exponents $\pm h_z$. $\vec{M}_\phi$ is a 2D map [cf. Fig. 1(b)] with Lyapunov exponents $\pm h_\phi$. The $\vec{\phi}$ motion is bounded.

The fixed point in $\vec{z}$ at $\vec{z} = 0$ (for all $\vec{\phi}$) corresponds to Λ. Thus $\delta\vec{M}_z(0, \vec{\phi}) = \delta\vec{M}_\phi(0, \vec{\phi}) = 0$. [In fact, $\delta\vec{M}_\phi$ is $O(|\vec{z}|)$, and by symmetry $\delta\vec{M}_z(\vec{z}, \vec{\phi})$ is $O(|\vec{z}|^3)$.] Since the dimension of SM in a region arbitrarily near Λ (i.e., for small $|\vec{z}|$) is the same as in any other region, we can neglect $\delta\vec{M}_z$ and $\delta\vec{M}_\phi$ for the purpose of calculating the dimension of SM. In that case we obtain

$$\vec{z}_{n+1} = DM_z(\vec{\phi}_n)\vec{z}_n, \;\; \vec{\phi}_{n+1} = \vec{M}_\phi(\vec{\phi}_n) \tag{3}$$

It is still not possible to analyze the system (3) for our billiard problem in a rigorous way. To proceed we therefore replace $DM_z(\vec{\phi})$ and $\vec{M}_\phi(\vec{\phi})$ by functional forms that are convenient for analysis and that preserve the basic structure of the billiard problem. We choose for $\vec{M}_\phi$ the cat map, $\vec{M}_\phi(\vec{\phi}) = C\vec{\phi}\,\mathrm{modulo}\,1$ where C is the cat map matrix ($C_{11} = 2$, $C_{12} = C_{21} = C_{22} = 1$), and for $DM_z(\vec{\phi})$ we choose

$$DM_z(\vec{\phi}) = \begin{bmatrix} \lambda & \sin 2\pi\phi \\ 0 & \lambda^{-1} \end{bmatrix}.$$

The Lyapunov exponents for this system, $\pm h_z$ and $\pm h_\phi$, are $h_z = \ln\lambda$ and $h_\phi = \ln B$, where B is the largest eigenvalue of C, namely $B = (\sqrt{5}+1)/2$. Vertical (parallel to z) line segments will be stretched by a factor of λ on each iterate. Thus, the distance from the invariant set to a nearby orbit increases by a factor of λ with each iterate. The exponential escape time is therefore given by $e^{-n/\tau} = \lambda^n$, or $\tau = (\ln\lambda)^{-1}$.

Since $\vec{z}$ is taken to evolve by the linear relationship $\vec{z}_{n+1} = DM_z(\vec{\phi})\vec{z}_n$, we have that for an orbit starting at a given $\vec{\phi}$, whether or not $|\vec{z}|$ decays exponentially toward Λ with time (i.e., $\vec{z}$ is on SM) depends on the orientation of $\vec{z}$ and not on $|\vec{z}|$. Thus the equation describing SM is of the form $z = z_s(\vec{\phi}, v_z) = f(\vec{\phi})v_z$. In particular, SM pictured in z versus v_z for some fixed value of $\vec{\phi}$ is a straight line passing through $z = v_z = 0$. (Note, however, that the function $f(\vec{\phi})$ can be very irregular causing SM to be fractal.)

To find $f(\vec{\phi})$ we define $\chi = z/v_z$, and substitute $z = \chi v_z$ into (3). We obtain $\chi_{n+1}v_{z_{n+1}} = \lambda\chi_n v_{z_n} + (\sin 2\pi\phi_n)v_{z_n}$, $v_{z_{n+1}} = \lambda^{-1}v_{z_n}$. Dividing the first equation by the second equation, $v_{z_{n+1}}$ and v_{z_n} cancel and we obtain for χ

$$\chi_{n+1} = \lambda^2\chi_n + \lambda\sin(2\pi\phi_n). \tag{4}$$

Equation (4), together with $\vec{\phi}_{n+1} = C\vec{\phi}_n\,\mathrm{modulo}\,1$, constitutes a three dimensional map system [as opposed to the four dimensional map system (3)]. For initial conditions in $\chi > f(\vec{\phi})$ ($\chi < f(\vec{\phi})$) we have that $\chi \to +\infty$ ($\chi \to -\infty$). Thus, if we iterate an initial condition backwards in time it tends to the surface $f(\vec{\phi})$. We make use of this to find $f(\vec{\phi})$. Imagine that we iterate from an initial $\vec{\phi}_0 = \vec{\phi}$ forward n steps to $C^n\vec{\phi}\,\mathrm{modulo}\,1$, choose χ_n, and then follow (4) backward to obtain the initial condition χ_0 that iterates to χ_n. Letting $n \to \infty$, and fixing χ_n, we have $\chi_0 \to f(\vec{\phi})$. This gives

$$f(\vec{\phi}) = -\lambda^{-1}\sum_{m=0}^{\infty}\lambda^{-2m}\sin[2\pi C^m\vec{\phi}], \tag{5}$$

which, for $(\ln\lambda^2)/\ln B = 2h_z/h_\phi < 1$ is known [12] to be a continuous, nowhere differentiable function (a Weierstraβ function of dimension $3 - 2h_z/h_\phi$). Furthermore, using the fact that $h_z = \tau^{-1}$ for this case, the dimension of the graph $\chi = f(\vec{\phi})$ is $3 - 2h_z/h_\phi$ [12], and the dimension of the graph of $z = f(\vec{\phi})v_z$ in the fully four dimensional phase space is thus given by (2).

We now conclude by discussing the possible experimental realization of our billiard system. Imagine placing a mirrored ellipsoid inside a tube (as shown in Fig. 1) which is mirrored on its inside. Here light rays play the role of orbits. They reflect from the mirrored surfaces in the same way that a test particle would bounce from a billiard (i.e., specularly). Imagine that the tube is oriented vertically with its bottom end (which is open) placed on a red surface and that the ellipsoid is suspended in the tube. In this configuration an observer looking in the top of the tube sees multiple reflections of the red surface that is at the bottom of the tube. Since rays whose directions are reversed retrace the same path, we can think of orbits as starting from the retina of the observer's eye (or film of a camera) passing through the pupil, bouncing around in the scatterer, and then exiting either through the bottom (red) or the top (not red). The boundary of the red region seen on the surface of the ellipsoid or the walls is precisely the basin boundary that we have discussed above. In preliminary work (with J.A. Yorke [13]) we have already applied this approach to a chaotic scatterer formed from four mirrored spheres to experimentally study a different general type of basin boundary structure known as a Wada boundary (a Wada boundary is a boundary separating three or more basins such that every boundary point is a boundary point for all basins).

There have been few experimental studies of basin boundaries because it is difficult to experimentally carry out the numerical technique of following orbits for many initial conditions chosen on a grid. The difficulty with this approach is that it is time consuming; it is often not possible to prepare initial conditions sufficiently precisely to observe small scale basin boundary structures, and experimental parameters may drift over the course of many runs. Nevertheless, in one case [6] this program was successfully carried out using an electronic circuit as the experimental system. In another work, Cusumano and Kimble [5] have formulated a new experimental procedure for studying basin boundaries which uses many initial conditions but not from a grid. To our knowledge, these two works and the four-sphere billiard discussed above, are so far the only experimental investigations of fractal basin boundaries. In contrast, there have been a large number of papers that have experimentally realized fractal stucture in chaotic attractors. This disparity in the situations of attracting and nonattracting chaotic sets seems to be largely due to the disparity in the ease of experimental realization for the two cases. We believe that optical billiard systems, like those described above, offer a convenient avenue for experimental investigation of the various general types of basin boundaries.

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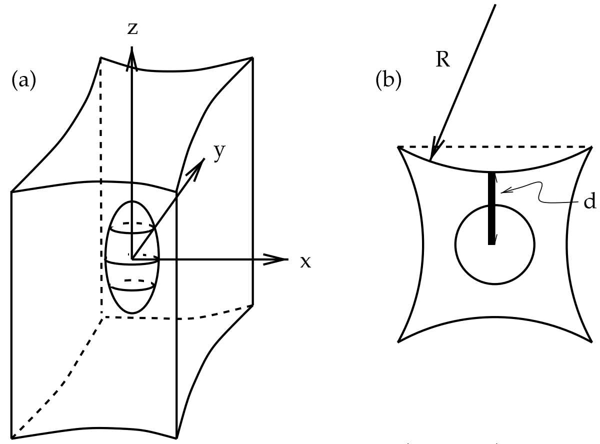

FIG. 1. The scattering system in the (atypical) case of vertical orientation of the ellipsoid. (a) A hard ellipsoid is placed inside a hard tube with cross-section as shown in (b). (The circle in (b) is the cross section of the ellipsoid.) $R = 25$, $d = 10$ and the radius of the ellipsoid at $z = 0$ is 5.

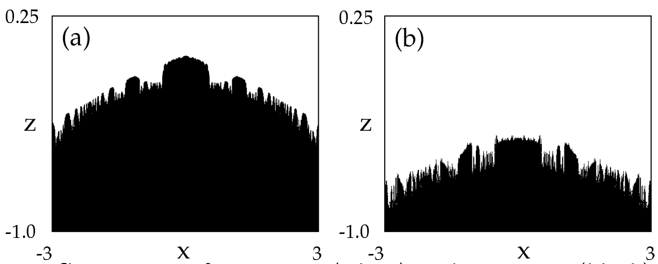

FIG. 2. Basins for $z \to \infty$ (white) and $z \to -\infty$ (black). $x\epsilon(-3, 3)$, $y = 5.1$, $z\epsilon(-2.5, 0)$, $v_x = 0$, $v_z = .1$,and v_y is given by the condition $|\vec{v}| = 1$ (numerical work was done in the 6D phase space) for (a) the untilted case and (b) a tilt of $\frac{2\pi}{100}$.

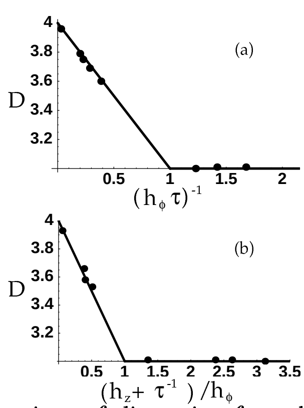

FIG. 3. Comparison of dimension formulae, Eq. (1) and Eq. (2), with numerical estimates (indicated by + and •).

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